UNIT 5

Limits and Rate of Change of Functions

Lesson 5.1: First, Some Background - Rational Functions

A family of functions that we are going to study, at this point, is the family of *rational functions*. A rational function is a function of the form

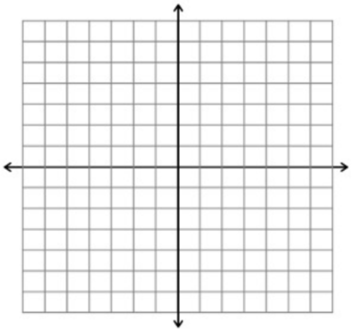
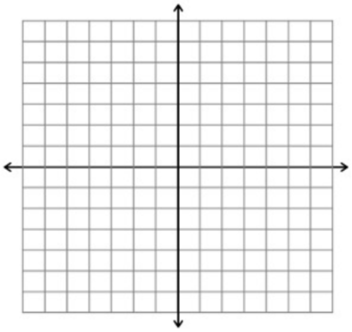
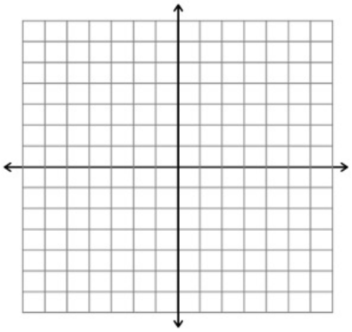
$$h(x) = \frac{f(x)}{g(x)}, \text{ where } f(x) \text{ and } g(x) \text{ are polynomials.}$$

One of the most important properties of rational functions is the presence of *asymptotes*, both *vertical and horizontal*. In the following exercises, we are going to try to deduce where these asymptotic "lines" appear.

> **Definition:** We define an *asymptote* as a line such that the distance between the graph of a curve and the line approaches zero as they tend to infinity.

Exploration 5.1.1: Rational Functions and Asymptotes

Sketch the graph for each function below. Sketch any vertical asymptotes as dashed lines. Fill in the missing information. Then make a hypothesis about where the rational function $h(x) = \dfrac{f(x)}{g(x)}$ will have vertical asymptotes.

1. $h(x) = \dfrac{1}{x-1}$  Vertical asymptote(s): Horizontal asymptote:	**2.** $h(x) = \dfrac{x+2}{x-4}$  Y-intercept: Horizontal asymptote:	**3.** $h(x) = \dfrac{x+6}{x^2-9}$  Vertical asymptote(s): X-intercept(s):

4. **Conjecture:** The rational function $h(x) = \dfrac{f(x)}{g(x)}$ will have vertical asymptotes at

   ____.

5. Use your conjecture to predict where the rational function

$$f(x) = \frac{x^2-9}{x-3}$$

   will have a vertical asymptote. Graph the function and see whether or not your conjecture is true. Create a rule for the discontinuity that occurs at $x = 3$ for this function.

6. Look at the three graphs that you drew in problems 1-3. See if you can articulate your findings about horizontal asymptotes, which can also be referred to as end behavior, and if you can explain why these rules apply. You will summarize your findings in two separate cases. The third case will be explored next.

7. Do you think that it is possible for the graph of a rational function to intersect its own horizontal asymptote? Why or why not? If you think that it is possible, see if you can find such a function.

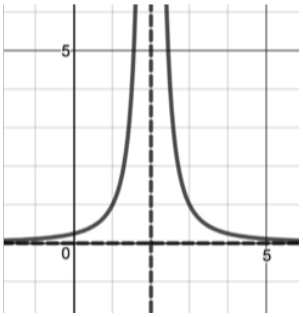
8. Do you think that it is possible for the graph of a rational function to intersect its own vertical asymptote? Why or why not? If you think that it is possible, see if you can find such a function.

Exploration 5.1.2: A Deeper Understanding of Rational Functions

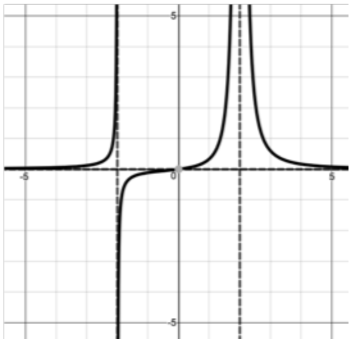
Part 1: Vertical Asymptotes

1. Sketch the graph of $y = \dfrac{1}{x-2}$, how would you describe the two portions of the graph with respect to each other?

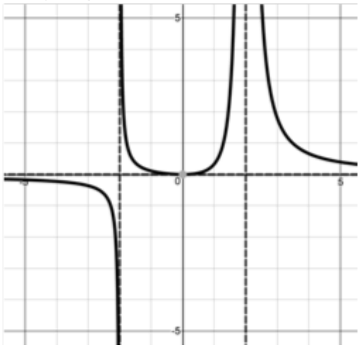
2. Create an equation for the graph of the following rational function. What are some differences between this function and the function from question 1?



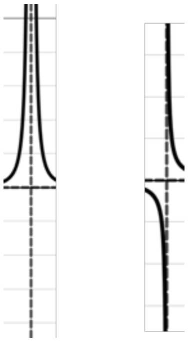
3. Write a possible function that could create the graph to the right. Explain your reasoning behind each part of the function.



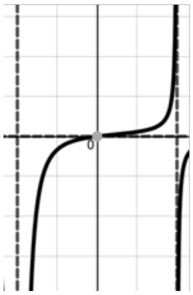
4. The function $y = \dfrac{x^2}{(x+2)(x-2)^2}$ creates the graph below. Identify which elements of the function affect the behaviors around the asymptotes.

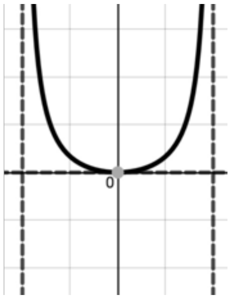


5. Given the two behaviors around a vertical asymptote, create a conjecture on what part of the function controls the behavior around the vertical asymptotes and how each could be generated.



6. Given two of the types of behavior between the vertical asymptotes, create a conjecture on what part of the function controls the behavior between the vertical asymptotes and how each could be generated.





Part 2: Slant Asymptotes

7. Perform the division of the rational function $f(x) = \dfrac{x^2 - x - 2}{x - 1}$ and sketch a graph. Make a conjecture about the equations for the asymptotes. Confirm your conjecture by analyzing the results of the function's division.

8. Sketch a graph for $f(x) = \dfrac{2x^2 - 6x + 3}{x - 3}$. Determine the equation of the slant asymptote.

Lesson 5.2: Limits

The concept of the *limit* of a function is a powerful tool that is used extensively in calculus. The limit enables one to describe much about the behavior of a function without actually graphing the function, even at values where the function does not exist. In fact, limits can describe the behavior of functions for *any* value of the independent variable including very large values.

A rather informal definition of the concept of *limit* is

> For a function $f(x),\ L$ is said to be the limit of $f(x)$ as x approaches a domain value c, if and only if, L is the one number that one can keep $f(x)$ arbitrarily close to just by keeping x close enough to c, but not equal to c.

The limit L of a function f as x approaches a number c is symbolized as

$$\lim_{x\to c} f(x) = L$$

In addition, the limit L of a function f as x approaches infinity is symbolized as

$$\lim_{x\to\infty} f(x) = L$$

The limit, L, if it exists, must be a number. Be aware that if $\lim\limits_{x\to c}$ goes off to infinity (by convention, this is symbolized: $\lim\limits_{x\to c} f(x) = \infty$), then the limit does not exist, because ∞ is not a number.

Suppose that the $\lim\limits_{x\to\infty} f(x) = a$ or $\lim\limits_{x\to-\infty} f(x) = a$. Then the horizontal line $y = a$ is called a horizontal asymptote of the graph of f.

As mentioned previously, part of the usefulness of the limit is that it allows one to get a feel for function behavior even at places where the function has a *discontinuity* or break in the graph. For example, consider the function

$$f(x) = \frac{(x-4)(x+5)}{(x-4)}$$

To find the $\lim\limits_{x\to 4} f(x)$, one should try direct substitution first by evaluating $f(x)$ at $x = 4$. A problem arises right away because the $f(4)$ yields the indeterminate form $\dfrac{0}{0}$. Thus, f has a discontinuity at $x = 4$. However, applying the limit allows one to cancel the $(x-4)$ term in the numerator and denominator of the function. Then the limit can be directly evaluated by substituting 4 in for x in the simplified limit statement. This is symbolized:

$$\lim_{x\to 4} \frac{\cancel{(x-4)}(x+5)}{\cancel{(x-4)}} = \lim_{x\to 4}(x+5) = 9$$

Exploration 5.2.1: Working with Limits

1. Graph $f(x) = \dfrac{(x-3)(x-9)}{(x-3)}$

2. Characterize what is happening with $f(x)$ at $x = 3$.

3. Explain how the value of the limit as x approaches 3 of $f(x)$ is connected to the graph of $f(x)$.

4. Graph $f(x) = \dfrac{1}{(x-5)}$ then make a conjecture as to the value or existence of $\lim\limits_{x \to 5} f(x)$.

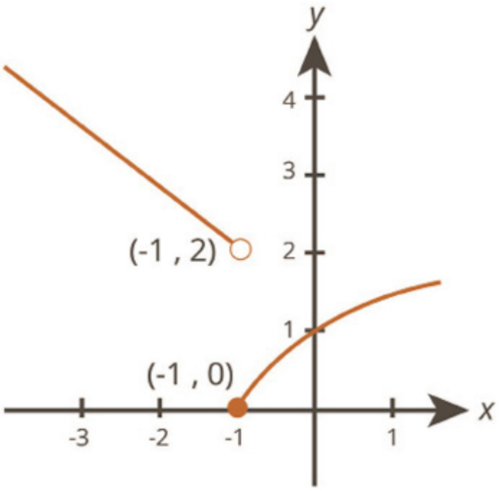
Help to verify your conjecture by filling in the provided table:

x	$f(x)$
4	
4.5	
4.9	
4.99	
5	
5.01	
5.1	
5.5	

What is the behavior of $f(x)$ as x approaches 5 from the positive or negative side?

5. Find all the vertical asymptotes or $f(x) = \dfrac{x}{\cos\, x}$. Characterize the limit of $f(x)$ as x approaches each of these vertical asymptotes.

6. Use the graph to determine the $\lim\limits_{x \to -1} f(x)$ (if any).



Find the limit if it exists:

7. $\lim\limits_{x \to 1} \dfrac{x^2 + x - 2}{x^2 - 3x + 2}$

8. $\lim\limits_{x \to 2} \dfrac{\frac{1}{x} - \frac{1}{2}}{x - 2}$

9. $\lim\limits_{x \to -2} (-x^2 + 4)$

[**HINT:** Factoring polynomials or finding a common denominator are techniques that often help one to evaluate the limit of a function, if it exists.]

Exploration 5.2.2: A Formal Definition of Limit

A formal definition of *the limit of a function* is:

$$L = \lim_{x \to c} f(x) \textit{ iff for any } \epsilon > 0 \textit{ there exists a number } \delta > 0$$

$$\textit{such that if } 0 < |x - c| < \delta \textit{ then } |f(x) - L| < \epsilon \, .$$

1. State this definition in your own words. Also identify the condition(s) given in the statement and what is concluded. [**HINT:** It will help you to understand what is being stated in the given definition if you recall what is meant by the absolute value of a quantity.]

2. On a pair of axes, precisely draw and label a picture that illustrates the meaning of the given formal definition of limit.

A summary of limit properties is provided in Table 5.2.1.

Table 5.2.1: Limit Properties as x Approaches a Constant a

Suppose that $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$ and k is some constant.

$$\lim_{x \to a} k \; f(x) = k \; \lim_{x \to a} f(x) = kL$$

$$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$$

$$\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M$$

$$\lim_{x \to a} (f(x)g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = LM$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} = \frac{L}{M}, \text{ if } M \text{ is not 0.}$$

Lesson 5.3: Approximating Rates of Change

We begin this Lesson with no background information. The next Exploration is intended to motivate your thoughts and to stimulate questions about the rate of change of functions. Recall that in Lesson 1.3, you investigated the rate of change of functions in a qualitative manner. In this section, we will investigate function rate of change from a quantitative standpoint.

In case you were wondering, the concept of limit will eventually play an important role in our investigation of rate of change.

Exploration 5.3.1: How fast is y changing?

A mass is oscillating on a spring while hanging from the ceiling (Figure 5.3.1-1). Its distance, y feet, from the ceiling is recorded using strobe photography every 1/10 sec. Consider movement away from the ceiling to be positive and movement toward the ceiling to be negative. The times and corresponding distances measured are:

t	y
0.2	3.99
0.3	5.84
0.4	7.37
0.5	8.00
0.6	7.48
0.7	6.01
0.8	4.16
0.9	2.63
1.0	2.00
1.1	2.52

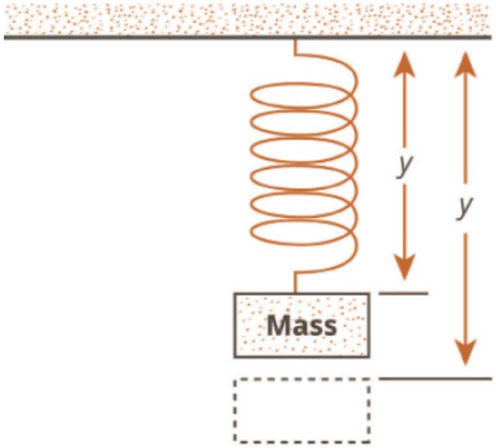


Figure 5.3.1-1

1. Devise a way to approximate how fast y is changing at each of the listed times.
    a. $t = 0.3$ sec.
    b. $t = 0.6$ sec.
    c. $t = 1.0$ sec.
2. Is there only one way to answer a., b., and c. of problem 1? Explain.
3. At time $t = 0.3$ sec, is the mass moving up or down? Justify your answer.

Exploration 5.3.1 should have you thinking about the difference between the *average rate of change* of a function and the *instantaneous rate of change* of a function. Exploration 5.3.2 should provide further insights into and some explanation about this subject.

Exploration 5.3.2: Difference Quotients and Rate of Change

In this Exploration we will use a situation modeled by an exponential function to investigate the rate of change of that function. There will be two different types of rate of change examined, the *average rate of change of the function and the instantaneous rate of change of the function.* The average rate of change of a function gives information about how the dependent variable of the function changes with respect to the independent variable between two independent values of the function. This can be expressed as the *average rate of change of* f, which is the quantity

$$\frac{f(x)-f(c)}{x-c}$$

where x is an independent value of choice and c is a particular value of interest in the domain of f. In what context have you encountered expression (1) in past mathematics classes?

Background

The algebraic quantity

$$\frac{f(x)-f(c)}{x-c}$$

is known as a *difference quotient*. We will eventually use difference quotients to help us define and find the *instantaneous rate of change of the function*. That is, the rate of change of a function f at one specific domain value of the function, namely $x = c$.

1. Is it possible to construct a difference quotient when $x = c$? Explain your answer.

Exploration

We will examine a car tire being punctured by a sharp object. As the air escapes from the tire, the distance d, in inches, between the rim of the tire and the street is a function of time t in seconds, where $t > 0$. From data collected, it was determined that this situation can be modeled by the exponential equation on the next page.

$$d(t) = 6(1.490^{-t})$$

Notice that this equation suggests that at $t = 0$, the rim was 6 inches above the ground.

Our goal is to estimate the rate of change of $d(t)$ at precisely the time $t = 3$ seconds. This will be accomplished by making use of the average rate of change and difference quotient concepts that were previously defined.

2. On graph paper, make a table of values and plot the graph of $d(t)$ for $t \geq 0$. $[$**Hint:** You may want to use a graphing calculator to compute the values.$]$

3. Find the average rate of change of $d(t)$ between the values (a) $t = 1$ sec. and $t = 3$ sec., between (b) $t = 2$ sec. and $t = 3$ sec., between (c) $t = 2.5$ sec. and $t = 3$ sec., and between (d) $t = 2.9$ sec., and $t = 3$ sec.

4. What do the average rates of change found in Exercise 3 represent with respect to the rim of the tire and its proximity to the street?

5. We want to find out how fast the rim of the tire is approaching the street at the instant $t = 3$ sec. Discuss how using the average rate of change is an approximation but not exact at the instant $t = 3$ sec.

The Connection to Limits

6. Notice that your answers for each interval supplied above seem to be approaching what is known as a *limit* or *limiting value*. Can you estimate this value?

7. What is the significance of the numerical sign of the limiting value in relation to the motion of the rim of the tire?

Historical Note:

The concept of the *limiting value* has been present in mathematics for over two thousand years. Archimedes (circa 250BC), mentioned in Lesson 4.4, used a limiting process to estimate the value of π . Both Isaac Newton (circa 1700) and Gottfried Leibniz (circa 1700 - Leibniz is also mentioned in Exploration 1.2.2) made use of the limit concept in each of their individual developments of the calculus. However, it was not until the mid-eighteen hundreds that Augustin-Louis Cauchy applied rigorous mathematical meanings to the concept of limit. Cauchy's $\epsilon - \delta$ definition of limit is still in standard use today.

The concept of *instantaneous rate of change* of a function was addressed and formalized chiefly by Gottfried Leibniz and Isaac Newton in their works relating to the development of the calculus. Other mathematicians of the time that contributed to this topic were Pierre de Fermat, René Descartes (also mentioned in Exploration 1.2.2 and the introduction to Unit 2), Christian Huygens, and Isaac Barrow. The development of this concept was born out of investigations involving continuous functions by these mathematicians having to do with the motion of objects, the tangent line problem, maximum-minimum problems, and area problems associated with non-standard shapes.

Back to Rates

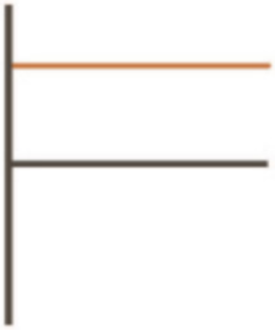
By considering specific velocity-time functions of the motion of a particle as rate of change functions and then trying to: 1) draw a rate of change function of the rate of change function and 2) draw a function that “undoes” the given rate of change function. Of course, we are talking about creating the acceleration-time function and the position-time functions, respectively, when supplied with a velocity-time function. As you work through Exploration 5.3.3, try to reconcile both the physical aspects and the mathematics theory behind your answers.

Exploration 5.3.3: An Advanced Qualitative Look at Rates

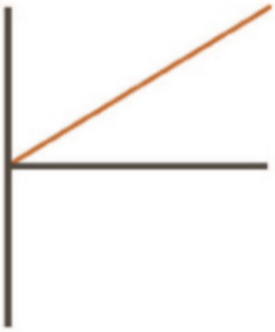
Work with gaining a further qualitative understanding of graphs of functions. In particular, focus on the relationships between acceleration, velocity and position vs. time in terms of rates of change of the functions given. Recall that velocity is the rate of change of position and acceleration is the rate of change of velocity.

For each of the following **velocity vs. time graphs** try to sketch what you think the corresponding *acceleration-time* graph will look like. Be prepared to present and justify your results.

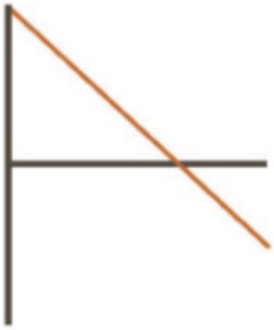
1.



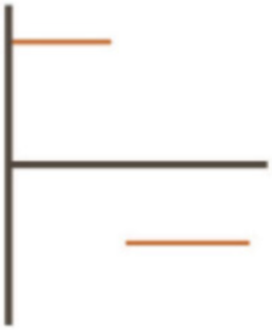
2.



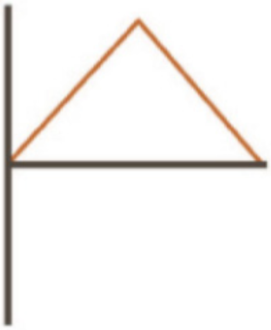
3.



4.



5.



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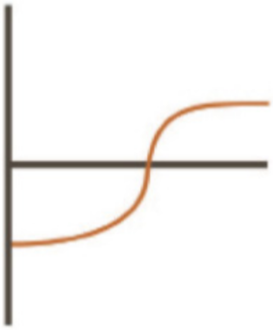

Challenge

As a challenge, for each of the above **velocity vs. time graphs** also try to sketch the corresponding *position-time* graph. [**HINT:** Think backwards.]

Lesson 5.4: The Derivative

Your work with the Explorations of Lessons 5.2 and 5.3 should have prepared you for an investigation of an important topic that you will investigate in detail when you take calculus, the *derivative of a function*. The derivative at a point P on a function f employs a limit to find the instantaneous rate of change of f at P.

Recall from Algebra I, first, that the *slope* of a linear function is equal to the rate of change in the independent variable (domain values) with respect to the change in the dependent variable (range values)

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}. \ (1)$$

This represents average rate of change.

> **Definition:** The derivative of a function f is the instantaneous rate of change of the dependent variable with respect to the independent variable. This is symbolized

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The derivative of a function is also a function. The derivative of a function evaluated at a certain domain value, x, of the function is a number, assuming the derivative is defined at x.

It is extremely important to notice that h is just a positive constant resulting in the fact that the quantity

$$\frac{f(x+h) - f(x)}{(x+h) - x}$$

is just a different way to write the slope formula (1) above that you have seen before. Thus a derivative is a limit applied to a slope for continuous functions on an interval. Exploration 5.3.2 illustrated the fact that an instantaneous rate of change of a function at a point can be found by finding the limiting value of the slopes of secant lines (representing average rates of change) as they become a tangent line at the point under investigation. Thus, the derivative of a function at a point is also the slope of the tangent line to the function at that point. This is illustrated in Figure 5.4-1.

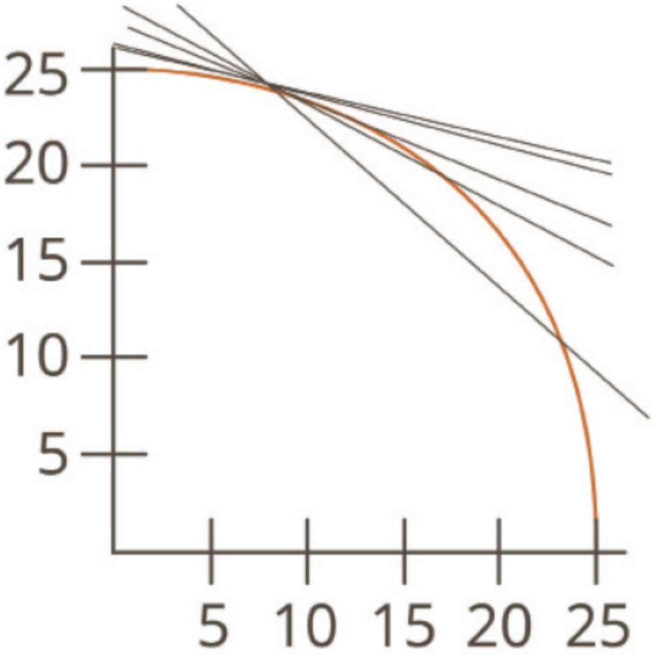

Figure 5.4-1: Secants Approaching a Tangent

For the purposes of this course, we will restrict our focus to applying derivatives to *polynomial* functions of the form

$$f(x) = a_0 + a_1x + a_2x^2.... + a_nx^n$$

Keep in mind, however, that derivatives can be applied to a wide array of functions, as you will see when you take calculus.

Let us consider the function $f(x) = x^2$. Applying the definition of the derivative of a function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

on $f(x) = x^2$, yields

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

Thus,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= 2x \end{aligned}$$

after simplification and application of the limit.

Quick Exercise: For $f(x) = x^2$, use the previous result to find:

$f'(0) =$ $\quad$ $f'(1) =$ $\quad$ $f'(-1) =$ $\quad$ $f'(3) =$

in general,

Power Rule: If $f(x) = x^n$, then $f'(x) = nx^{n-1}$

For our purposes, we will consider that n in the power rule is a rational number. The proof of this, however, is beyond the scope of this course. We also note that, since the derivative of a function is a limit, it follows the limit properties of Table 5.2.1. Thus, we can extend the derivative power rule to any polynomial of the form

$$f(x) = a_0 + a_1x + a_2x^2 .... + a_nx^n$$

by differentiating (that is, applying the derivative) term-by-term within the polynomial $f(x)$.

Exploration 5.4.1: Applying the Power Rule

Find $f'(x)$.

1. $f(x) = x^5$

2. $f(x) = 7x^3$

3. $f(x) = \dfrac{1}{x^2}$

4. $f(x) = \sqrt{x}$

5. $f(x) = 5x^3 - 4x^2 - 7$

6. $f(x) = 5$

7. A projectile is shot upward with an initial velocity of 96 ft/sec. The equation for the height of the projectile after t seconds is $h(t) = -16t^2 + 96t + 256$.

a. The velocity function (or rate of change of the height) is the derivative of $h(t)$. Find this derivative function.

b. Is the velocity of the projectile increasing or decreasing after 1 second?

c. If the acceleration of the projectile is the rate of change of the velocity function, what is the acceleration function for the projectile?

8. Newton's Law of Gravitation states that the gravitational force F between two objects of mass m_1 and m_2 is

$$F = Gm_1m_2r^{-2}, \text{ where } r > 0$$

and r represents the distance between objects. G is a gravitational constant.

a. Find the rate of change of F between the two objects.

b. Discuss why the *sign* of your answer to (a.) makes sense.

Exploration 5.4.2: The Derivative and Function Information

Recall, from Unit 1, that you were supplied with the definitions of an increasing function and a decreasing function. This definition is repeated here:

Definition: If a function $y = f(x)$ is defined on an interval $[a, b]$, then $f(x)$ is *increasing on* $[a, b]$, if for any two points in [a,b], whenever $x_1 < x_2$, then $f(x_1) < f(x_2)$. The function $f(x)$ is *decreasing on* $[a, b]$, if for any two points in [a,b], whenever $x_1 < x_2$ then $f(x_1) > f(x_2)$.

1. Based on this definition and your work in this Unit, can you make a conjecture about the relationship between the *sign* (positive or negative) of the value of the derivative of a function at a given point and whether the function is increasing or decreasing at that point?

2. If a continuous function attains a maximum or minimum value on a given interval, what conclusion, if any, can you make about the value of the derivative at the maximum or minimum? Justify your answer.

3. For the quadratic $y = ax^2 + bx + c$, use your answer to problem 2 to help you decide what the value of x will be for the vertex point of the quadratic. Is this value familiar to you?